Rigid Fluid: Animating The Interplay Between Rigid Bodies And Fluid

(Notes From The Paper)

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(The section numbers, equation numbers and references in these notes are the same as those in the original paper to ease cross-referencing.)

1. Introduction

Fluids and solids can have 3 kinds of coupling:

1. *One way solid-to-fluid coupling*

   Ex: A heavy ball which splashes a puddle of water.

2. *One way fluid-to-solid coupling*

   Ex: A cork floating on top of a flowing stream of water.

3. *Two way coupling of solids and fluids*

   Ex: A lead ball sinking in a pool of water.

This paper will try to solve (3.). (3.) can be seen as a superset that encompasses both (1.) and (2.). The paper also solves the interaction between the rigid bodies themselves when they're inside the fluid.

2. Previous Work

- **Takahashi et al. [2002]**: Two-way coupling of buoyant rigid bodies and incompressible fluids using a combined *Volume Of Fluid* and *Cubic Interpolated Propagation System*. Using a grid, any cell that is more than half filled with a rigid body is identified as a *solid boundary*. Zero *Neumann boundary conditions* are set for the pressure at these boundaries to approximate solid-to-fluid coupling.

  Neumann boundary conditions:

  (http://en.wikipedia.org/wiki/Neumann_boundary_conditions)

  Imposed on an ordinary or partial differential equation, it specifies the values the *derivative* of a solution is to take on the boundary of the *domain*.

- **Takahashi et al. [2003]**: Achieve a solid-to-fluid coupling by setting the velocity of the fluid inside a cell containing a solid to that of the solid.

  Both of the above techniques model forces due to hydrostatic pressure, but neglect the dynamic forces and torques caused by fluid momentum.

- **Génevaux et al. [2003]**: Two-way coupling between an incompressible fluid and deformable solids modeled using *mass-spring systems*, with a communication interface between the two. Technique cannot be used for non-deformable rigid bodies.

3. Equations Of Motion For Fluid

Navier-Stokes equations describe the motion of a viscous incompressible fluid:

$$(1)\text{: } \nabla.\boldsymbol{u} = 0$$

$$(2)\text{: } \boldsymbol{u_t} = -(\boldsymbol{u}.\nabla)\boldsymbol{u} + \nabla.(\nu\nabla\boldsymbol{u}) - \frac{1}{\rho}\nabla p + \boldsymbol{f}$$

Conservation Of Mass

Eq (1) represents the *conservation of mass* for fluids.

$\boldsymbol{u}$: Fluid velocity

∇: Called *del* or *nabla* or *gradient*. It is the vector of spatial partial derivatives.

In 3 dimensions, it is given as:

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

When ∇ is applied on a scalar field f, it results in a *gradient field*:

$$\nabla.f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

$\nabla.f$ is also known as *divergence*. If $\nabla.f = 0$, then it is a *divergenceless field*.

The *continuity equation* for the conservation of mass for fluids is written as:

(http://scienceworld.wolfram.com/physics/ContinuityEquation.html)

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\boldsymbol{u}) = 0$$

ρ: Fluid density

For an incompressible fluid, ρ is constant. So, $\frac{\partial\rho}{\partial t} = 0$. Hence, (1) holds true.

Conservation Of Momentum

Eq (2) represents the *conservation of momentum* for fluids.

$\boldsymbol{u_t}$: Time derivative of fluid velocity, ie., $\boldsymbol{u_t} = \frac{\partial\boldsymbol{u}}{\partial t}$

ν: Kinematic viscosity

p: Scalar pressure field

$\boldsymbol{f}$: Body force per unit mass. Usually, gravity.

Eq (2) is of the form:

$$\frac{\text{momentum}}{\text{mass}} = \text{mass} + \text{viscosity} + \text{pressure} + \text{body force}$$

Solving Navier-Stokes Equations

Step 1: Best Guess Velocity

Solve for a *best guess velocity ($\tilde{\boldsymbol{u}}$)* without taking pressure into account. The pressure value is not yet known at this point. Also, this is called a best guess value because $\tilde{\boldsymbol{u}}$ is *not* divergence free.

$$(3):\ \tilde{\boldsymbol{u}} = \boldsymbol{u} + \Delta t[-(\boldsymbol{u}.\nabla)\boldsymbol{u} + \nabla.(\nu\nabla\boldsymbol{u}) + \boldsymbol{f}]$$

Step 2: Pressure Projection

Next is *pressure projection*. The pressure term left out of (3) is accounted for by:

$$(5):\ \boldsymbol{u}^{\text{new}} = \tilde{\boldsymbol{u}} - \frac{\Delta t}{\rho}\nabla p$$

If $\boldsymbol{u}^{\text{new}}$ is the correct velocity, it should satisfy the incompressibility constraint. That is:

$$(\bar{5}):\ \nabla.\boldsymbol{u}^{\text{new}} = 0$$

Taking the divergence of (5), we get:

$$\nabla.\boldsymbol{u}^{\text{new}} = \nabla.\tilde{\boldsymbol{u}} - \nabla.(\frac{\Delta t}{\rho}\nabla p)$$

Using ($\bar{5}$) the above results in:

$$(7):\ \nabla.\tilde{\boldsymbol{u}} = \frac{\Delta t}{\rho}\nabla^2 p$$

Now, p is the only unknown in (7). So, the equation can be solved and p can be calculated. This p is then substituted back in (5) to obtain $\boldsymbol{u}^{\text{new}}$.

4. Rigid Bodies

$\mathbb{F}$: *Fluid domain*

$\mathbb{R}$: *Rigid body domain*

$\mathbb{F}$ and $\mathbb{R}$ are disjoint. They have a *shared boundary* $\partial\mathbb{R}$.

$\mathbb{C}$: *Computational domain*. $\mathbb{C} = \mathbb{F} \bigcup \mathbb{R}$

Rigidity Constraint

A rigid body complies to the *rigidity constraint*, i.e. it is both divergence free (conserves mass) and deformation free (conserves shape). This is enforced using a *Lagrange multiplier*. For every point y_j in a rigid body, the following should hold:

$$(8):\ \dot{y}_j = v + \omega \times r_j$$

$\dot{y}_j$: Velocity at y_j

r_j: Vector from x, the center of mass of the rigid body to the point y_j

v: Translational velocity at x

ω: Rotational velocity about x

Eq (8) can be seen as an equation that represents the motion of a rigid body as "translations and rotations about its center of mass".

The above representation won't be used in this paper. Instead, a deformation operator is used.

Deformation Operator

The rigidity constraint can also be represented by means of a *deformation operator* $\boldsymbol{D}$. It is defined for any vector field $\boldsymbol{u} = (u, v, w)$ as:

$$(9)\text{: } \boldsymbol{D}[\boldsymbol{u}] = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T]$$

Applying ∇ on $\boldsymbol{u}$ gives:

$$\nabla.\boldsymbol{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

Using $u_x = \frac{\partial u}{\partial x}$ and so on, we get:

$$\nabla.\boldsymbol{u} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Using the above result in (9) gives:

$$\boldsymbol{D}[\boldsymbol{u}] = \frac{1}{2}\begin{pmatrix} 2u_x & (v_x + u_y) & (w_x + u_z) \\ (v_x + u_y) & 2v_y & (w_y + v_z) \\ (w_x + u_z) & (w_y + v_z) & 2w_z \end{pmatrix}$$

The 3x3 symmetric tensor $\boldsymbol{D}[\boldsymbol{u}]$ resulting above represents the *spatial deformation* of $\boldsymbol{u}$.

For rigid body motion to satisfy the rigidity constraint, the following should hold:

$$(10)\text{: } \boldsymbol{D}[\boldsymbol{u}] = \boldsymbol{0} \text{ in } \mathbb{R}$$

5. Governing Equations

Assumptions:

- There is no *viscoelastic stress*. The fluid has viscosity, but no elasticity.
- All boundary conditions except those between the rigid body and the fluid are understood.

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The *conservation of momentum equation for fluids*:

$$(11)\text{: } \boldsymbol{u_t} = -(\boldsymbol{u}.\nabla)\boldsymbol{u} + \nabla.(\nu\nabla\boldsymbol{u}) - \frac{1}{\rho_f}\nabla p + \boldsymbol{f} \text{ in } \mathbb{F}$$

ρ_f: Mass density of the fluid.

The *conservation of momentum equation for a rigid body*:

$$(12)\text{: } \boldsymbol{u_t} = -(\boldsymbol{u}.\nabla)\boldsymbol{u} + \nabla.\Pi - \frac{1}{\rho_r}\nabla p + \boldsymbol{f} \text{ in } \mathbb{R}$$

ρ_r: Rigid body density.

The viscosity term is missing since it doesn't apply to solids. Π represents the *deformation stress* in the body required to maintain the rigidity.

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Deformation constraint is *stronger* than the divergence constraint and it holds for rigid bodies in Eq (10)

and

divergence constraint holds for fluids in Eq (1)

hence

the divergence constraint holds for the entire computational domain.

That is:

$$(13)\text{: } \nabla.\boldsymbol{u} = 0 \text{ in } \mathbb{C}$$

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The *no-slip boundary condition* for the force of the fluid on the solid is:

$$(14)\text{: } \boldsymbol{u} = \boldsymbol{u}_i \text{ on } \partial\mathbb{R}$$

$\boldsymbol{u}_i$: Velocity on $\partial\mathbb{R}$

The *dynamic force boundary condition* for the force of the fluid on the solid is:

$$(14)\text{: } (2\rho_f \nu \boldsymbol{D}[\boldsymbol{u}] - p\mathbf{1}).\boldsymbol{n} = t \text{ on } \partial\mathbb{R}$$

$\mathbf{1}$: Identity tensor

$\boldsymbol{n}$: Normal on $\partial\mathbb{R}$

$\boldsymbol{t}$: Traction force of the fluid on the solid.

(14) represents the traction force as a "sum of projected viscous stress and pressure".

Similar equations can be written for the force of solids on fluids.

6. Implementation Of Governing Equations

$$\boldsymbol{u}^n \rightarrow \boldsymbol{u}^{n+1}$$

is broken into

$$\boldsymbol{u}^n \rightarrow \boldsymbol{u}^* \rightarrow \hat{\boldsymbol{u}} \rightarrow \boldsymbol{u}^{n+1}$$

6.1. Solving Navier-Stokes Equations: $\boldsymbol{u}^n \rightarrow \boldsymbol{u}^*$

Solve $\boldsymbol{u}^n$ in $\mathbb{C}$ in 4 steps:

1. Add the body force $\Delta t\boldsymbol{f}$ to all of $\mathbb{C}$.

   Since rigid body motion is enforced only inside $\mathbb{R}$, there will be a *slip error* at $\partial\mathbb{R}$.

   The slip error is proportional to $|\rho_r - \rho_f|$.

   The error will be handled later.

2. Solve the advection term using Stam's technique.

3. Solve the diffusion term using the *Implicity Variable Viscosity Formulation*.

4. Use pressure projection to make the velocity in $\mathbb{C}$ divergence free.

Result: Divergence free velocity field $\boldsymbol{u}^*$ in $\mathbb{C}$.

6.2. Calculating Rigid Body Forces: $\boldsymbol{u}^* \rightarrow \hat{\boldsymbol{u}}$

Collision forces are included in the velocity field to transfer momentum between the solid and fluid domains.

Maintain the *sum of accelerations* on a rigid body due to collision forces.

$$(15): \boldsymbol{A}_c = \sum_j \frac{\boldsymbol{F}_j}{M_i}$$

$\boldsymbol{A}_c$: Sum of accelerations caused due to collision forces on a rigid body.

$\boldsymbol{F}_j$: A collision force.

$\boldsymbol{M}_i$: Mass of the rigid body.

$i \epsilon \{1, 2, ..., N\}, N$ is the number of rigid bodies

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Maintain the *sum of angular accelerations* about the center of mass of rigid body caused due to collision forces.

$$(16): \alpha_c = \boldsymbol{\Sigma} \boldsymbol{I}_i^{-1}[(\boldsymbol{p}_j - \boldsymbol{x}_i) \times \boldsymbol{F}_j]$$

α_c: Sum of angular accelerations about center of mass of a rigid body.

$\boldsymbol{I}_i$: Moment of inertia of the i^{th} rigid body.

$\boldsymbol{p}_j$: Point of application of force $\boldsymbol{F}_j$.

$\boldsymbol{x}_i$: Center of mass of the i^{th} rigid body.

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Forces that arise due to *relative density* $(\frac{\rho_r}{\rho_f})$ are also considered. As the relative density increases, it becomes difficult for the fluid to move a rigid body.

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Together, the collision forces and the relative density are accounted for in $\mathbb{R}$ by introducing a *source term*:

$$(17): \boldsymbol{S} = \rho_r \boldsymbol{A}_c + \boldsymbol{r}_i \times \rho_r \alpha_c - (\rho_r - \rho_f)[\frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} + (\boldsymbol{u}^*.\nabla)\boldsymbol{u}^* - \boldsymbol{f}]$$

$\boldsymbol{r}_i$: $(y_i - x_i)$, vector from center of mass to grid point location in $\mathbb{R}_i$

Eq (17) can be solved since all the required values are known.

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Using S, we solve for the new velocity field:

$$(18): \hat{\boldsymbol{u}} = \boldsymbol{u}^* + w\frac{\Delta t}{\rho_r}\boldsymbol{S}$$

w: Fraction of the computational cell occupied by solid. 0 for fluid, 1 for solid.

6.3. Enforcing Rigid Motion: $\hat{\boldsymbol{u}} \rightarrow \boldsymbol{u}^{n+1}$

We know that final velocity formulation is of the form:

$$(19): \boldsymbol{u}^{n+1} = \hat{\boldsymbol{u}} + \frac{\Delta t}{\rho_r}\boldsymbol{R}$$

$\boldsymbol{R}$: The unknown force that maintains rigidity.

Rigidity is described by the deformation operator. Hence by replacing (19) in (10):

$$(20)\colon \boldsymbol{D}[\boldsymbol{u}^{n+1}] = \boldsymbol{D}[\hat{\boldsymbol{u}} + \tfrac{\Delta t}{\rho_r}\boldsymbol{R}] = 0$$

$\boldsymbol{R}$ is found from (20) and substituted in (19) to obtain $\boldsymbol{u}^{n+1}$.

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Alternatively, $\hat{\boldsymbol{u}}$ in $\mathbb{R}$ can be broken into 2 parts:

$$(21)\colon \hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_R + \hat{\boldsymbol{u}}'$$

where,

$\hat{\boldsymbol{u}}_R$ is the rigid body velocity we are searching for

$\hat{\boldsymbol{u}}'$ is the stress force inside $\mathbb{R}$ that enforces rigid body motion upon it. This is given by:

$$(22)\colon \hat{\boldsymbol{u}}' = -\frac{\Delta t}{\rho_r}\boldsymbol{R}$$

The desired rigid body solution of (20) and (22) for $\boldsymbol{R}$ and $\hat{\boldsymbol{u}}'$ must conserve momentum. Hence, it can be obtained directly.

The union of (8) over each rigid body gives:

$$(23)\colon \hat{\boldsymbol{u}}_R = \bigcup_i (\hat{\boldsymbol{v}}_i + \hat{\omega}_i \times r_i)$$

Because momentum is conserved, each rigid body's $\hat{v}_i$ and $\hat{\omega}_i$ can be obtained directly by integrating the intermediate $\hat{\boldsymbol{u}}$ inside the rigid body $\mathbb{R}_i$ using the following equations:

$$(24)\colon M_i \hat{v}_i = \int_{\mathbb{R}_i} \rho_i \hat{\boldsymbol{u}}\, \mathrm{dg}_i$$

$$(25)\colon I_i \hat{\omega}_i = \int_{\mathbb{R}_i} r_i \times \rho_i \hat{\boldsymbol{u}}\, \mathrm{dg}_i$$

where,

M_i is the mass of the i^{th} rigid body

I_i is the moment of inertia of the i^{th} rigid body

ρ_i is the density of the i^{th} rigid body

dg_i is the volume of the grid cell occupied by the i^{th} rigid body

(24) and (25) are evaluated by summing the appropriate values for each grid cell in $\mathbb{R}_i$.

We now use results of (24) and (25) to get the rigid body velocity $\hat{\boldsymbol{u}}_R$. This is then distributed over the rigid bodies to get the final velocity:

$$(26)\colon \boldsymbol{u}^{n+1} = (1-w)\hat{\boldsymbol{u}} + w\, \hat{\boldsymbol{u}}_R$$

This equation enforces rigidity and conserves momentum inside $\mathbb{R}$.

7. Advancing The Computational Domain

Particle level set is used for $\mathbb{F}$ and a rigid body solver is used for $\mathbb{R}$.